

ELEMENTARY PROOFS OF THE GAUSS-BONNET THEOREM AND OTHER INTEGRAL FORMULAS IN \mathbb{R}^3

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ABSTRACT. For a compact differentiable surface with boundary embedded in \mathbb{R}^3 , we give simple proofs of the Gauss-Bonnet theorem, Poincaré-Hopf theorem, and several other integral formulas. We complete all of the proofs without using fundamental or differential forms.

1. INTRODUCTION

In this note we prove several integral formulas involving the principal curvatures of a surface, including the Gauss-Bonnet theorem. All of the proofs are elementary in the sense that we do not place any co-ordinates on the surface, and hence we do not need to invoke the theory of fundamental or differential forms. The results will follow from applying Stokes' theorem directly to the surface, and studying objects associated with the surface that are co-ordinate invariant. In particular, the principal curvatures of the surface will arise as the directional derivatives of the normal vector to the surface.

2. PRELIMINARIES

All of the identities we will prove are for a compact, oriented C^∞ surface M with boundary ∂M embedded in \mathbb{R}^3 with volume element dA . At each point on M we denote the position vector by \mathbf{X} and the normal vector by \mathbf{N} . Our results will follow from this variation of Stokes' Theorem:

Claim. *Let f and g be differentiable functions that map an \mathbb{R}^3 -neighborhood of M to \mathbb{R} . If \mathbf{P} and \mathbf{Q} are orthonormal vector fields on M that are not necessarily continuous, and are oriented so that $\mathbf{P} \times \mathbf{Q} = \mathbf{N}$ everywhere on M , then:*

$$\int_{\partial M} f dg = \int_M [(\nabla_{\mathbf{P}} f)(\nabla_{\mathbf{Q}} g) - (\nabla_{\mathbf{Q}} f)(\nabla_{\mathbf{P}} g)] dA$$

Proof. We start with:

$$\int_{\partial M} f dg = \int_{\partial M} f \nabla g \cdot d\mathbf{X}.$$

Using Stokes' Theorem and the identity:

$$\text{curl}(f\nabla g) = \nabla f \times \nabla g$$

gives:

$$\begin{aligned}\int_{\partial M} f dg &= \int_M \nabla f \times \nabla g \cdot \mathbf{N} dA \\ &= \int_M \nabla f \cdot \nabla g \times \mathbf{N} dA.\end{aligned}$$

Since \mathbf{P} and \mathbf{Q} form an orthonormal basis for the tangent plane at any point on M , any tangent vector can be written as the sum of its projection onto \mathbf{P} and its projection onto \mathbf{Q} . Therefore:

$$\begin{aligned}\nabla g \times \mathbf{N} &= (\nabla g \times \mathbf{N} \cdot \mathbf{P})\mathbf{P} + (\nabla g \times \mathbf{N} \cdot \mathbf{Q})\mathbf{Q} \\ &= (\nabla g \cdot \mathbf{N} \times \mathbf{P})\mathbf{P} + (\nabla g \cdot \mathbf{N} \times \mathbf{Q})\mathbf{Q} \\ &= (\nabla g \cdot \mathbf{Q})\mathbf{P} - (\nabla g \cdot \mathbf{P})\mathbf{Q} \\ &= (\nabla_{\mathbf{Q}} g)\mathbf{P} - (\nabla_{\mathbf{P}} g)\mathbf{Q}.\end{aligned}$$

We finally get:

$$\begin{aligned}\int_{\partial M} f dg &= \int_M [(\nabla_{\mathbf{Q}} g)(\nabla f \cdot \mathbf{P}) - (\nabla_{\mathbf{P}} g)(\nabla f \cdot \mathbf{Q})] dA \\ &= \int_M [(\nabla_{\mathbf{P}} f)(\nabla_{\mathbf{Q}} g) - (\nabla_{\mathbf{Q}} f)(\nabla_{\mathbf{P}} g)] dA.\end{aligned}$$

□

In what follows we will assume that \mathbf{P} and \mathbf{Q} are orthonormal principal directions on M with corresponding principal curvatures κ_1 and κ_2 , and that they are oriented so that $\mathbf{P} \times \mathbf{Q} = \mathbf{N}$ everywhere on M . We will use the subscripts p and q to denote the directional derivatives of a function in the directions \mathbf{P} and \mathbf{Q} , respectively. An immediate corollary of the above claim is that if \mathbf{V} and \mathbf{W} are vector fields on \mathbb{R}^3 then:

$$(1) \quad \int_{\partial M} \mathbf{V} \cdot d\mathbf{W} = \int_M (\mathbf{V}_p \cdot \mathbf{W}_q - \mathbf{V}_q \cdot \mathbf{W}_p) dA.$$

Since \mathbf{P} and \mathbf{Q} are not required to be continuous in the above equation, we will not run into difficulties if M contains umbilical points. With these definitions we have:

$$(2) \quad \mathbf{N}_p = -\kappa_1 \mathbf{P}, \quad \mathbf{N}_q = -\kappa_2 \mathbf{Q}.$$

Finally, we define the mean curvature H and Gaussian curvature K at points on M by:

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1 \kappa_2.$$

3. TWO CURVATURE IDENTITIES

We begin with two simple consequences of identity (1):

Claim. *If \mathbf{V} is a vector field on \mathbb{R}^3 then:*

$$(3) \quad \int_{\partial M} (\mathbf{V} \times \mathbf{N}) \cdot d\mathbf{X} = - \int_M (\mathbf{V}_p \cdot \mathbf{P} + \mathbf{V}_q \cdot \mathbf{Q} + 2H \mathbf{V} \cdot \mathbf{N}) dA$$

$$(4) \quad \int_{\partial M} (\mathbf{V} \times \mathbf{N}) \cdot d\mathbf{N} = \int_M (\kappa_2 \mathbf{V}_p \cdot \mathbf{P} + \kappa_1 \mathbf{V}_q \cdot \mathbf{Q} + 2K \mathbf{V} \cdot \mathbf{N}) dA.$$

Proof. Using equations (1) and (2) we get:

$$\begin{aligned}
& \int_{\partial M} (\mathbf{V} \times \mathbf{N}) \cdot d\mathbf{X} \\
&= \int_M (\mathbf{V}_p \times \mathbf{N} \cdot \mathbf{Q} - \kappa_1 \mathbf{V} \times \mathbf{P} \cdot \mathbf{Q} - \mathbf{V}_q \times \mathbf{N} \cdot \mathbf{P} + \kappa_2 \mathbf{V} \times \mathbf{Q} \cdot \mathbf{P}) dA \\
&= \int_M (\mathbf{V}_p \cdot \mathbf{N} \times \mathbf{Q} - \kappa_1 \mathbf{V} \cdot \mathbf{P} \times \mathbf{Q} - \mathbf{V}_q \cdot \mathbf{N} \times \mathbf{P} + \kappa_2 \mathbf{V} \cdot \mathbf{Q} \times \mathbf{P}) dA \\
&= \int_M (-\mathbf{V}_p \cdot \mathbf{P} - \mathbf{V}_q \cdot \mathbf{Q} - 2H \mathbf{V} \cdot \mathbf{N}) dA \\
& \quad \int_{\partial M} (\mathbf{V} \times \mathbf{N}) \cdot d\mathbf{N} \\
&= \int_M (-\kappa_2 \mathbf{V}_p \times \mathbf{N} \cdot \mathbf{Q} + \kappa_1 \kappa_2 \mathbf{V} \times \mathbf{P} \cdot \mathbf{Q} + \kappa_1 \mathbf{V}_q \times \mathbf{N} \cdot \mathbf{P} - \kappa_1 \kappa_2 \mathbf{V} \times \mathbf{Q} \cdot \mathbf{P}) dA \\
&= \int_M (-\kappa_2 \mathbf{V}_p \cdot \mathbf{N} \times \mathbf{Q} + \kappa_1 \kappa_2 \mathbf{V} \cdot \mathbf{P} \times \mathbf{Q} + \kappa_1 \mathbf{V}_q \cdot \mathbf{N} \times \mathbf{P} - \kappa_1 \kappa_2 \mathbf{V} \cdot \mathbf{Q} \times \mathbf{P}) dA \\
&= \int_M (\kappa_2 \mathbf{V}_p \cdot \mathbf{P} + \kappa_1 \mathbf{V}_q \cdot \mathbf{Q} + 2K \mathbf{V} \cdot \mathbf{N}) dA.
\end{aligned}$$

□

Equation (3) is a generalization of the divergence theorem for surfaces to vector fields on \mathbb{R}^3 that appears for example in §7 of [2]. With these two identities we can quickly prove several well-known results.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the standard basis for \mathbb{R}^3 . Successively setting \mathbf{V} equal to these basis elements in (3) and (4) gives the vector identities:

$$\begin{aligned}
\int_{\partial M} \mathbf{N} \times d\mathbf{X} &= -2 \int_M H \mathbf{N} dA \\
\int_{\partial M} \mathbf{N} \times d\mathbf{N} &= 2 \int_M K \mathbf{N} dA.
\end{aligned}$$

Setting \mathbf{V} equal to the vectors $\mathbf{e}_1 \times \mathbf{X}$, $\mathbf{e}_2 \times \mathbf{X}$ and $\mathbf{e}_3 \times \mathbf{X}$ in (3) and (4) gives the vector identities:

$$\begin{aligned}
\int_{\partial M} \mathbf{X} \times (\mathbf{N} \times d\mathbf{X}) &= -2 \int_M H \mathbf{X} \times \mathbf{N} dA \\
\int_{\partial M} \mathbf{X} \times (\mathbf{N} \times d\mathbf{N}) &= 2 \int_M K \mathbf{X} \times \mathbf{N} dA.
\end{aligned}$$

Setting $\mathbf{V} = \mathbf{X}$ in (3) and (4) gives Minkowski's formulas (cf. [3], pp. 181-185):

$$\begin{aligned}
\int_{\partial M} \mathbf{X} \times \mathbf{N} \cdot d\mathbf{X} &= -2 \int_M (1 + H \mathbf{X} \cdot \mathbf{N}) dA \\
\int_{\partial M} \mathbf{X} \times \mathbf{N} \cdot d\mathbf{N} &= 2 \int_M (H + K \mathbf{X} \cdot \mathbf{N}) dA.
\end{aligned}$$

4. THE GAUSS-BONNET AND POINCARÉ-HOPF THEOREMS

We next prove the Gauss-Bonnet theorem and the Poincaré-Hopf theorem using (4) and (1). We start with the following simplified version of Liouville's formula (cf. [1], problem 11.19):

Claim. Suppose that there exists a constant vector \mathbf{C} with $\|\mathbf{C}\| = 1$ such that $\mathbf{C} \cdot \mathbf{N} \neq \pm 1$ on ∂M . Let s denote arc length, with the subscript s denoting the derivative with respect to arc length. Let θ denote the angle between the unit tangent vector \mathbf{X}_s to ∂M and the vector $\mathbf{C} \times \mathbf{N}$ tangent to M . There on ∂M there holds:

$$(5) \quad \theta_s = \kappa_g - \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} \mathbf{C} \times \mathbf{N} \cdot \mathbf{N}_s$$

where κ_g is the geodesic curvature of ∂M .

Proof. We use $[]$ to denote the triple product of vectors:

$$[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3] = \mathbf{v}_1 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_3 = \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3).$$

We have the relations:

$$\mathbf{X}_s \cdot \mathbf{X}_s = 1, \quad \mathbf{X}_s \cdot \mathbf{N} = 0, \quad \mathbf{N} \cdot \mathbf{N} = 0.$$

Differentiating these with respect to s gives:

$$\mathbf{X}_s \cdot \mathbf{X}_{ss} = 0, \quad \mathbf{X}_{ss} \cdot \mathbf{N} = -\mathbf{X}_s \cdot \mathbf{N}_s, \quad \mathbf{N}_s \cdot \mathbf{N} = 0.$$

The geodesic curvature κ_g is defined by:

$$\kappa_g = [\mathbf{X}_s \mathbf{X}_{ss} \mathbf{N}]$$

and the angle θ satisfies the relation:

$$\tan \theta = \frac{\mathbf{X}_s \cdot \mathbf{C}}{[\mathbf{X}_s \mathbf{C} \mathbf{N}]}.$$

Differentiating $\tan^{-1} \theta$ gives:

$$\theta_s = \frac{[\mathbf{X}_s \mathbf{C} \mathbf{N}](\mathbf{X}_{ss} \cdot \mathbf{C}) - (\mathbf{X}_s \cdot \mathbf{C})([\mathbf{X}_{ss} \mathbf{C} \mathbf{N}] + [\mathbf{X}_s \mathbf{C} \mathbf{N}_s])}{[\mathbf{X}_s \mathbf{C} \mathbf{N}]^2 + (\mathbf{X}_s \cdot \mathbf{C})^2}.$$

By projecting onto the orthonormal basis $\{\mathbf{X}_s, \mathbf{N}, \mathbf{X}_s \times \mathbf{N}\}$ for \mathbb{R}^3 we get:

$$\begin{aligned} \mathbf{X}_{ss} &= -(\mathbf{X}_s \cdot \mathbf{N}_s)\mathbf{N} - \kappa_g \mathbf{X}_s \times \mathbf{N} \\ \mathbf{N}_s &= (\mathbf{X}_s \cdot \mathbf{N}_s)\mathbf{X}_s + [\mathbf{N}_s \mathbf{X}_s \mathbf{N}] \mathbf{X}_s \times \mathbf{N}, \quad \mathbf{N}_s \times \mathbf{X}_s = [\mathbf{N}_s \mathbf{X}_s \mathbf{N}] \mathbf{N} \\ \|\mathbf{C}\|^2 &= (\mathbf{C} \cdot \mathbf{X}_s)^2 + (\mathbf{C} \cdot \mathbf{N})^2 + [\mathbf{C} \cdot \mathbf{X}_s \times \mathbf{N}]^2 = 1. \end{aligned}$$

Therefore:

$$\begin{aligned} \theta_s &= \frac{[\mathbf{X}_s \mathbf{C} \mathbf{N}](\kappa_g [\mathbf{X}_s \mathbf{C} \mathbf{N}] - (\mathbf{C} \cdot \mathbf{N})(\mathbf{X}_s \cdot \mathbf{N}_s))}{[\mathbf{X}_s \mathbf{C} \mathbf{N}]^2 + (\mathbf{X}_s \cdot \mathbf{C})^2} \\ &\quad + \frac{\kappa_g (\mathbf{X}_s \cdot \mathbf{C})^2 - (\mathbf{X}_s \cdot \mathbf{C})(\mathbf{C} \cdot \mathbf{N}_s \times \mathbf{X}_s)}{[\mathbf{X}_s \mathbf{C} \mathbf{N}]^2 + (\mathbf{X}_s \cdot \mathbf{C})^2} \\ &= \kappa_g - \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} ([\mathbf{X}_s \mathbf{C} \mathbf{N}](\mathbf{X}_s \cdot \mathbf{N}_s) + (\mathbf{X}_s \cdot \mathbf{C})[\mathbf{N}_s \mathbf{X}_s \mathbf{N}]) \\ &= \kappa_g - \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} ((\mathbf{X}_s \times \mathbf{N}) \cdot ((\mathbf{X}_s \cdot \mathbf{C})\mathbf{N}_s - (\mathbf{X}_s \cdot \mathbf{N}_s)\mathbf{C})) \\ &= \kappa_g - \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} ((\mathbf{X}_s \times \mathbf{N}) \cdot (\mathbf{X}_s \times (\mathbf{N}_s \times \mathbf{C}))) \\ &= \kappa_g - \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} [\mathbf{C} \mathbf{N} \mathbf{N}_s] \end{aligned}$$

□

We can now prove:

$$(6) \quad \int_{\partial M} \kappa_g ds + \int_M K dA = 2\pi\chi(M)$$

We integrate (5) over the boundary of each triangle. As in the classical proof, the θ_s term in (5) gives rise to the Euler characteristic term $\chi(M)$ when the exterior angles at the vertices of each triangle are enumerated. To derive the surface integral we substitute:

$$\mathbf{V} = \frac{\mathbf{C} \cdot \mathbf{N}}{1 - (\mathbf{C} \cdot \mathbf{N})^2} \mathbf{C}$$

$$\frac{1 + (\mathbf{C} \cdot \mathbf{N})^2}{(1 - (\mathbf{C} \cdot \mathbf{N})^2)^2} [(\mathbf{C} \cdot \mathbf{P})^2 + (\mathbf{C} \cdot \mathbf{Q})^2] K - \frac{2(\mathbf{C} \cdot \mathbf{N})^2}{1 - (\mathbf{C} \cdot \mathbf{N})^2} K,$$
$$\|\mathbf{C}\|^2 = (\mathbf{C} \cdot \mathbf{P})^2 + (\mathbf{C} \cdot \mathbf{Q})^2 + (\mathbf{C} \cdot \mathbf{N})^2 = 1.$$

□

Claim. *If \mathbf{V} is a vector field on M with $\|\mathbf{V}\| = 1$ then:*

$$\int_{\partial M} \mathbf{V} \times \mathbf{N} \cdot d\mathbf{V} = \int_M K dA$$

$$\begin{aligned} \int_{\partial M} \mathbf{V} \times \mathbf{N} \cdot d\mathbf{V} \\ = \int_M (-\kappa_1 \mathbf{V} \times \mathbf{P} \cdot \mathbf{V}_q + \kappa_2 \mathbf{V} \times \mathbf{Q} \cdot \mathbf{V}_p - 2 \mathbf{V}_p \times \mathbf{V}_q \cdot \mathbf{N}) dA. \end{aligned}$$
$$\mathbf{V}_p \cdot \mathbf{V} = \mathbf{V}_q \cdot \mathbf{V} = \mathbf{N} \cdot \mathbf{V} = 0,$$
$$\mathbf{V} \times \mathbf{P} = -(\mathbf{V} \cdot \mathbf{Q})\mathbf{N}, \quad \mathbf{V} \times \mathbf{Q} = (\mathbf{V} \cdot \mathbf{P})\mathbf{N}$$
$$\mathbf{V}_p \cdot \mathbf{N} = \kappa_1 \mathbf{V} \cdot \mathbf{P}, \quad \mathbf{V}_q \cdot \mathbf{N} = \kappa_2 \mathbf{V} \cdot \mathbf{Q}.$$
$$\kappa_1 \kappa_2 [(\mathbf{V} \cdot \mathbf{P})^2 + (\mathbf{V} \cdot \mathbf{Q})^2] = K.$$

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Corollary. *If \mathbf{V} is a vector field on M that does not vanish on ∂M and has only isolated singularities on M at points in the set S then:*

$$\int_{\partial M} \frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} = \int_M K dA - 2\pi \sum_{s \in S} \text{Index}_s(\mathbf{V})$$

Proof. This follows from:

$$\frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} = \frac{\mathbf{V}}{\|\mathbf{V}\|} \times \mathbf{N} \cdot d\left(\frac{\mathbf{V}}{\|\mathbf{V}\|}\right)$$

and the fact that near singularities s of \mathbf{V} , the boundary integral approaches $-d\theta$ for the vector field projected onto the tangent plane to M at s . \square

Combining this with the Gauss-Bonnet theorem to eliminate the total curvature term gives:

Theorem (Poincaré-Hopf). *If \mathbf{V} is a vector field on M satisfying the same conditions as in the above corollary then:*

$$\chi(M) - \sum_{s \in S} \text{Index}_s(\mathbf{V}) = \frac{1}{2\pi} \int_{\partial M} \left(\frac{\mathbf{V} \times \mathbf{N}}{\|\mathbf{V}\|^2} \cdot d\mathbf{V} + \kappa_g ds \right).$$

The above result extends the Poincaré-Hopf theorem to vector fields that are not perpendicular to ∂M . When \mathbf{V} is perpendicular to ∂M , the boundary integral reduces to zero and gives the traditional result.

5. IDENTITIES CONTAINING THE DIFFERENCE OF PRINCIPAL CURVATURES

We conclude by proving two identities that contain a term equal to the difference of the principal curvatures, instead of their sum or product:

Claim. *For any twice differentiable function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ there holds:*

$$\begin{aligned} \int_{\partial M} \nabla F(\mathbf{X}) \cdot d\mathbf{N} &= - \int_M (\kappa_2 - \kappa_1) [(\nabla^2 F(\mathbf{X}))] \mathbf{P} \cdot \mathbf{Q} \\ \int_{\partial M} \nabla F(\mathbf{N}) \cdot d\mathbf{X} &= \int_M (\kappa_2 - \kappa_1) [(\nabla^2 F(\mathbf{N}))] \mathbf{P} \cdot \mathbf{Q}. \end{aligned}$$

Proof. These follow from (1) and:

$$\begin{aligned} \nabla F(\mathbf{X})_p &= (\nabla^2 F(\mathbf{X})) \mathbf{P} & \nabla F(\mathbf{N})_p &= -\kappa_1 (\nabla^2 F(\mathbf{N})) \mathbf{P} \\ \nabla F(\mathbf{X})_q &= (\nabla^2 F(\mathbf{X})) \mathbf{Q} & \nabla F(\mathbf{N})_q &= -\kappa_2 (\nabla^2 F(\mathbf{N})) \mathbf{Q}. \end{aligned}$$

\square

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